

Embedding Quantum Information into Classical Spacetime: Perspective to Tsallis Statistics and AdS/CFT Correspondence

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We rigorously prove that the nonextensive parameter in Tsallis statistics for d -dimensional maximally entangled quantum states is equivalent to the square of the curvature radius of $(d+1)$ -dimensional asymptotically anti-de Sitter (AdS) spacetime. We argue a unique role of AdS metric on efficiently embedding complex quantum data into classical information space.

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Tsallis statistics [1] is one-parameter extension of standard one by defining the Tsallis nonextensive entropy

$$S_q = \frac{1 - \sum_{i=1}^m \lambda_i^q}{q-1}, \quad (1)$$

where $\sum_{i=1}^m \lambda_i = 1$. This entropy approaches the von Neumann entropy in the $q \rightarrow 1$ limit

$$S_1 = -\lim_{q \rightarrow 1} \sum_{i=1}^m \lambda_i^q \ln \lambda_i = -\sum_{i=1}^m \lambda_i \ln \lambda_i. \quad (2)$$

When we consider a part of the total quantum system, S_1 is called entanglement entropy, and a set of λ_i is given by the eigenvalues of a density matrix tracing over environmental degrees of freedom. It has been extensively examined that the q -deformed statistical distribution appears in many complex systems [2, 3].

The Tsallis entropy comes from thermally equilibrium state where a partial system is combined with environment with heat capacity $1/(q-1)$ [4]. Actually, S_q saturates at the value $1/(q-1)$ for $q > 1$ if we take $m \rightarrow \infty$. Thus the parameter q controls how efficiently the environment works as a heat bath. Since the partial system can not reach at thermal equilibrium instantaneously, it is hard to access all of the physical states in our phase space. That is the origin for nonergodicity and metastable states of complex systems. In other words, q changes the amount of thermal fluctuation between the system and the environment, and then we see various course-grained systems labeled by q .

This situation is very similar to that of holographic theories, such as anti-de Sitter space / conformal field theory (AdS/CFT) correspondence that is one of central problems in superstring theory [5] and multiscale entanglement renormalization ansatz (MERA) developed recently in statistical physics [6]. In those cases, the fundamental problem is to understand how we distribute amount of original quantum information in a spatially d -dimensional (dD) system into $(d+1)$ -dimensional classical

spacetime. The extra dimension in the classical spacetime of AdS/CFT and MERA represents a renormalization flow parameter of the original quantum system. Such a function of the extra dimension may have some correspondence to the nonextensive parameter q in Tsallis statistics. In particular, statistical distributions similar to Eq. (1) frequently appears in multifractal systems. The scaling properties of the fractal would be closely related to the conformal invariance in CFT. Complexity is in some sense complementary to the presence of scaling properties, since the complexity itself is originated from strong competition between states with various energy and length scales. Therefore, it is very interesting to give a unified viewpoint to those problems.

Motivated by the similarity, we examine geometrical structure of classical information space into which the original quantum information given by Eq. (1) is embedded [8, 9]. The geometry is a really crucial factor to determine the total capacity of the information space. In connection with AdS/CFT and MERA, a question is whether the hyperbolic (AdS) geometry is automatically induced from a very fundamental physical and mathematical level. The purpose of this Letter is to answer this question.

In order to connect Tsallis entropy with geometric quantities, it is useful to introduce relative Tsallis entropy, since the relative entropy can be regarded as a measure of 'distance' between two points (probability distributions) in our phase space (we call it information space). The relative von Neumann entropy between two probability distributions $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and $\Lambda = \{\Lambda_1, \Lambda_2, \dots, \Lambda_m\}$, is usually defined by

$$V(\lambda, \Lambda) = \sum_{i=1}^m \lambda_i \ln \frac{\lambda_i}{\Lambda_i}. \quad (3)$$

This is called Kullback-Liebert measure or divergence. We examine q -extension of this quantity, $V_q(\lambda, \Lambda)$, but before going to the main discussion, we first summarize basic properties of this simplest case.

Let us introduce the difference between λ and Λ as

$$\Lambda_i = \lambda_i + \epsilon_i, \quad (4)$$

where we assume $\sum_{i=1}^m \epsilon_i = 0$.

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We expand $V(\lambda, \Lambda)$ by the small parameter ϵ

$$\begin{aligned} V(\lambda, \Lambda) &= -\sum_i \lambda_i \ln \left(1 + \frac{\epsilon_i}{\lambda_i} \right) \\ &= -\sum_i \lambda_i \left\{ \frac{\epsilon_i}{\lambda_i} - \frac{1}{2} \left(\frac{\epsilon_i}{\lambda_i} \right)^2 + \dots \right\} \\ &= \frac{1}{2} \sum_i \lambda_i \left(\frac{\epsilon_i}{\lambda_i} \right)^2 + \dots \end{aligned} \quad (5)$$

We find that $V(\lambda, \Lambda)$ is always positive, and is an increasing function with respect to ϵ . Thus we can regard $V(\lambda, \Lambda)$ as a measure of distance between λ and Λ in our information space.

A key ingredient in this study is to consider that the probability distribution function contains a set of $(d+1)$ continuous parameters x^0, x^1, \dots, x^d :

$$\lambda_i = \lambda_i(x^0, x^1, \dots, x^d), \quad (6)$$

$$\Lambda_i = \lambda_i(x^0 + dx^0, x^1 + dx^1, \dots, x^d + dx^d), \quad (7)$$

where dx^μ is an infinitesimal parameter. According to MERA, the left hand side of each equation is 'quantum' representation, while the right hand side is 'classical' one. The parametric representation is truly a continuous version of the tensor contraction in the MERA network. Therefore, we expect that the geometry appears in the right hand side. We expand the logarithm of Λ_i upto the second order as

$$\ln \Lambda_i = \ln \lambda_i + \frac{\partial \ln \lambda_i}{\partial x^\mu} dx^\mu + \frac{1}{2} \frac{\partial^2 \ln \lambda_i}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu, \quad (8)$$

where the Einstein's convention is used. Multiplying λ_i into both sides of Eq. (8) and summing upto all i , we obtain

$$\begin{aligned} \sum_i \lambda_i \ln \Lambda_i &= \sum_i \lambda_i \ln \lambda_i + \sum_i \frac{\partial \lambda_i}{\partial x^\mu} dx^\mu \\ &\quad + \sum_i \lambda_i \frac{1}{2} \frac{\partial^2 \ln \lambda_i}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu. \end{aligned} \quad (9)$$

Here the second term of the right hand side vanishes. Then we find

$$\sum_i \lambda_i \ln \frac{\Lambda_i}{\lambda_i} = -\frac{1}{2} \sum_i \lambda_i \frac{\partial^2 \ln \lambda_i}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu. \quad (10)$$

By using the following relation

$$\frac{\partial^2 \ln \lambda_i}{\partial x^\mu \partial x^\nu} = -\frac{\partial \ln \lambda_i}{\partial x^\mu} \frac{\partial \ln \lambda_i}{\partial x^\nu} + \frac{1}{\lambda_i} \frac{\partial^2 \lambda_i}{\partial x^\mu \partial x^\nu}, \quad (11)$$

we obtain the following representaton

$$V(\lambda, \Lambda) = \frac{1}{2} g_{\mu\nu} dx^\mu dx^\nu, \quad (12)$$

where $g_{\mu\nu}$ is Fisher's information matrix defined by

$$g_{\mu\nu} = \sum_{i=1}^m \lambda_i \frac{\partial \ln \lambda_i}{\partial x^\mu} \frac{\partial \ln \lambda_i}{\partial x^\nu} = \sum_{i=1}^m \frac{1}{\lambda_i} \frac{\partial \lambda_i}{\partial x^\mu} \frac{\partial \lambda_i}{\partial x^\nu}. \quad (13)$$

This is nothing but a metric tensor of our information space in terms of general relativity.

Let us move to evaluation of relative Tsallis entropy

$$V_q(\lambda, \Lambda) = \frac{1}{q-1} \left\{ \sum_{i=1}^m \lambda_i \left(\frac{\Lambda_i}{\lambda_i} \right)^{q-1} - 1 \right\} \quad (14)$$

$$= \frac{1}{q-1} \left(\sum_{i=1}^m \lambda_i^q \Lambda_i^{1-q} - 1 \right). \quad (15)$$

This form is determined so that $V_q(\lambda, \Lambda)$ approaches $-S_q$ for $\Lambda_i = 1$ and approaches zero for $\Lambda_i = \lambda_i$.

First we confirm whether this can be regarded as a distance in our information space:

$$\left(\frac{\Lambda_i}{\lambda_i} \right)^{q-1} = \left(1 + \frac{\epsilon_i}{\lambda_i} \right)^{1-q} \quad (16)$$

$$\simeq 1 + (1-q) \left(\frac{\epsilon_i}{\lambda_i} \right) - \frac{1}{2} q(1-q) \left(\frac{\epsilon_i}{\lambda_i} \right)^2 \quad (17)$$

Then

$$V_q(\lambda, \Lambda) = \frac{1}{2} q \sum_{i=1}^m \lambda_i \left(\frac{\epsilon_i}{\lambda_i} \right)^2 + \dots \geq 0. \quad (18)$$

This result is direct q -extension of Eq. (5).

We expand Λ_i^{1-q} by dx^μ as follows

$$\begin{aligned} \Lambda_i^{1-q} &= \lambda_i^{1-q} + \frac{\partial \lambda_i^{1-q}}{\partial x^\mu} dx^\mu + \frac{1}{2} \frac{\partial^2 \lambda_i^{1-q}}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu \\ &= \lambda_i^{1-q} + (1-q) \lambda_i^{-q} \frac{\partial \lambda_i}{\partial x^\mu} dx^\mu \\ &\quad - \frac{1}{2} q(1-q) \lambda_i^{-1-q} \frac{\partial \lambda_i}{\partial x^\mu} \frac{\partial \lambda_i}{\partial x^\nu} dx^\mu dx^\nu \\ &\quad + \frac{1}{2} (1-q) \lambda_i^{-q} \frac{\partial^2 \lambda_i}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu. \end{aligned} \quad (19)$$

Then we obtain

$$V_q(\lambda, \Lambda) = \frac{1}{2} g_{\mu\nu}(q) dx^\mu dx^\nu, \quad (21)$$

$$g_{\mu\nu}(q) = q \sum_{i=1}^m \frac{1}{\lambda_i} \frac{\partial \lambda_i}{\partial x^\mu} \frac{\partial \lambda_i}{\partial x^\nu} = q g_{\mu\nu}(1). \quad (22)$$

Clearly, q characterizes Weyl transformation in string theory. Thus, we expect the presence of hidden conformal symmetry. In terms of metric, the q -deformation is very simple. This means that the essential geometric structure of the information space is invariant by the deformation, although the deformation changes Boltzmann's exponential distribution into power-law behavior. It should be noted that this feature does not clearly

appear in Rényi entropy. This sharp difference may be originated from stability of the information space [10, 11].

Let us consider geometrical and physical meaning of the metric tensor $g_{\mu\nu}(q)$. We compare $g_{\mu\nu}(q)$ with the AdS metric given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{R^2}{z^2} \left(dz^2 + \sum_{i=1}^d dx_i^2 \right), \quad (23)$$

where R is curvature radius, $\eta_{\mu\nu} dx^\nu = dx_\mu$, and we denote radial axis as $x^0 = z$. It is no doubt correct except for the factor $1/2$ to relate a positive q value with R as

$$q = R^2, \quad (24)$$

if $g_{\mu\nu}(q)$ is converted into AdS. Actually, the case of $q > 1$ properly represents coarse graining by environmental fluctuation.

In density matrix renormalization group (DMRG) calculations and in optimizing matrix product states, it has been shown for various quantum 1D and classical 2D models that all of λ_i plays a role on statistical properties of our target system at quantum criticality. For simplicity, we consider the maximally entangled case given by

$$\lambda_i = \frac{1}{m}. \quad (25)$$

Then the metric is represented as

$$g_{\mu\nu}(q) = q \frac{1}{m^2} \frac{\partial m}{\partial x^\mu} \frac{\partial m}{\partial x^\nu}. \quad (26)$$

In the case of Eq. (25), the entanglement entropy for $q = 1$ is given by $S_1 = \ln m$.

For 1D quantum lattice cases, we can state the physical meaning of Eq. (25) more specifically. The entanglement entropy ($q = 1$) near criticality obeys the finite-entanglement scaling [12–14] (a discrete version of Calabrese-Cardy CFT formula)

$$S_1 = \frac{c\kappa}{6} \ln m, \quad (27)$$

where c is the central charge, and the finite-entanglement scaling exponent κ is defined by

$$\kappa = \frac{6}{c(\sqrt{12/c} + 1)}. \quad (28)$$

Thus, our discussion is exact in the $c \rightarrow \infty$ limit at least for $q = 1$. This condition is also compatible to a necessary condition of AdS/CFT for which the quantum effect on the gravity is neglected.

Hereafter we present a method for solving a set of differential equations in 1D cases, and we denote the coordinate as $(x^0, x^1) = (z, x)$. However, the generalization to higher dimensions is straightforward. First, we compare the g_{zx} -component of AdS with that of Eq. (26):

$$\frac{1}{z^2} = \frac{1}{m(z, x)^2} \frac{\partial m(z, x)}{\partial z} \frac{\partial m(z, x)}{\partial x}. \quad (29)$$

This condition is equal to the following couple of differential equations

$$\frac{\partial m(z, x)}{\partial z} = \pm \frac{m(z, x)}{z}. \quad (30)$$

They have two solutions, but the better one is

$$m(z, x) = \frac{f(x)}{z}, \quad (31)$$

since m monotonically decreases with increasing z , and finally the system reaches at a fixed point.

Next, we compare the g_{xx} -component of AdS with that of Eq. (26):

$$\frac{1}{z^2} = \frac{1}{m(z, x)^2} \frac{\partial m(z, x)}{\partial x} \frac{\partial m(z, x)}{\partial x}. \quad (32)$$

This leads to the following differential equations

$$\frac{\partial m(z, x)}{\partial x} = \pm \frac{m(z, x)}{z}, \quad (33)$$

and the solutions are given by

$$m(z, x) = m^* e^{\pm x/z} = m^* \left(1 \pm \frac{x}{z} + \dots \right). \quad (34)$$

where m^* is the fixed point value of m . This is basically consistent with Eqs. (30) and (31) for large z .

We also compare the g_{zx} -component of AdS with that of Eq. (26):

$$0 = \frac{1}{m(z, x)^2} \frac{\partial m(z, x)}{\partial z} \frac{\partial m(z, x)}{\partial x}. \quad (35)$$

Substituting Eq. (34) into this equation, we find that Eq. (34) satisfies this equation for large z .

As a result, we obtain the following asymptotic forms of metric and 'holographic' eigenvalues of the density matrix in the 'IR' limit (not UV):

$$2V_q = \frac{q}{z^2} \left(dz^2 + \sum_{i=1}^d dx_i^2 \right), \quad (36)$$

$$m(z, x) = m^* \exp \left(\frac{1}{z} \sum_{i=1}^d |x_i| \right), \quad (37)$$

$$\lambda_i(z, x) = \frac{1}{m^*} \exp \left(-\frac{1}{z} \sum_{i=1}^d |x_i| \right). \quad (38)$$

In the IR limit, $m(z, x) \rightarrow 1$, and actually this feature has been observed in our recent paper [15]. In 1D cases, $c = 3\sqrt{q}/2G$ (G is Newton constant) [16].

Let us discuss implications of the present result. In particular, we would like to construct a classical viewpoint of our information space. According to the Ryu-Takayanagi formula for the holographic entanglement entropy [17], the original quantum data are decomposed into a set of data with different length scales, and each

of them is embedded into different layers of the holographic space with AdS metric. This can be visualized by MERA [6]. In those cases, the entropy is derived by calculating the geodesic minimal length connecting two end points of a partial system with length L in the original quantum system. Then, each layer behaves as a rather 'classical' system. There the total holographic entropy can be obtained by summing up all of contributions from each layer, but the entropy in each layer strictly satisfies the area law scaling that is required for gapped systems. Our result Eq. (38) clearly shows this situation: λ is a kind of a correlation function, and shows exponential decay for the spatial coordinate, suggesting that each holographic layer is gapped. The decay rate is given by the radial coordinate z , and when we go to IR region, the decay becomes much slower than the UV region.

The most important key factor would be the assumption given by Eq. (25). As we have already mentioned, all of the quantum information are important at criticality, and various energy and length scales are incorporated. Except for very specific models that have abnormal energy spectra, general interacting quantum models should have various energy scales, since the hierarchy of the Heisenberg equations of motion for field operators of fundamental excitations do not terminate within finite orders. This seems to be very strong constraint for the entanglement scaling. In my naive image, the hyperbolic space is technically very convenient, since we can represent various length scales in a unified way by only using single metric. However, the present result indicates that the nature necessarily selects this spacetime.

In my recent work on connecting DMRG-based approaches with image processing, it has been observed numerically that a snapshot of the 2D classical Ising model ($c = 1/2$) at critical temperature can be decomposed into a set of patterns by the singular value decomposition, and the data are embedded into a 3D hyperbolic space [18].

In this case also, the IR limit of the holographic entanglement entropy for the patterns obeys a clear scaling relation compatible to the finite entanglement entropy scaling in Eq. (27). This scaling also appears in more general cases, such as zebra image and flower image [19]. In those cases, the original images do not have conformal invariance, but, after course graining realized by dropping small singular values and corresponding eigenstates, the hyperbolic space and the entanglement scaling automatically emerge from the data. In other words, we drop a part of very fine data that are not directly associated with CFT. The presence of the hidden hyperbolic space seems to be a common background in real-space RG.

It has been well known that various power-law behaviors in complex systems can be systematically described by the q -deformed statistics. In our sense, the complexity itself is crucial for the q -deformation. Furthermore, it is recently observed that the randomness seems to be very far from conformal symmetry but they are sometimes complementary [20]. They indicate the presence of hidden AdS geometry in those problems. This indication is also related to the problem hidden in the zebra image in Ref. [19]. In addition, the Parisi solution with replica symmetry breaking in spin glass has also similar eigenvalue spectra, and the multifractality and complex free energy structure due to the breaking would have the same origin to the above problems.

Summarizing, we have found that the Tsallis relative entropy for maximally quantum entanglement systems can be converted into a line element in one-higher dimensional classical AdS space. Then, the nonextensive parameter in the Tsallis entropy is equivalent to the square of curvature radius of asymptotically anti-de Sitter spacetime. We have explained a role of the holographic eigenvalues on embedding the quantum data into classical AdS space. That is a core factor of the physics of AdS/CFT and MERA in an information-theoretical viewpoint.

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